SPATIAL CONSTRAINTS ON GERRYMANDERING:
A PRACTICAL COMPARISON OF METHODS

JAMES SAXON
Harris School of Public Policy and Center for Spatial Data Science, University of Chicago

ABSTRACT. In the half-century since the US Supreme Court asserted its dominion over legislative districting, it has recognized the harm of political gerrymandering but failed to provide relief for it. An earlier, extended period of successful Congressional action suggests reviving the legislative remedy. Historically, Congress required equipopulous, contiguous, and compact districts, but the formal definition of compactness has proven contentious. Does it matter?

This paper presents a credible, quantitative framework for evaluating the implications of spatial constraints in districting reform. By implementing a flexible automated districting procedure with eighteen different definitions of compactness, I evaluate their practical implications: the seat shares that they imply for the two parties and for racial and ethnic minorities. On these grounds, the definitions are markedly consistent. The choice among compactness definitions need not be contentious.

Political gerrymandering, the manipulation of the borders of legislative districts for partisan gain, affects party representation and reduces voter confidence in government. The United States will next redistrict in 2020-2022 after the 24th Census, and gerrymandering stands as a fulcrum in the balance of national political power.

Despite a pervasive focus in this domain on the courts, judicial intervention appears unlikely. It has been half a century since the “Reapportionment Revolution” when the Supreme Court assumed responsibility for enforcing the equipopulation of legislative districts. Three decades have passed since the Court ruled in Davis v. Bandemer (478 U.S. 109, 1986) that political gerrymandering is a justiciable harm. But the Court has failed in that time to identify a “clear and manageable standard” or a “limited and precise rationale” for recognizing gerrymandering, and has declined to provide relief. (Vieth v. Jubelirer, 541 U.S. 267, 2004) In the 2017-2018 term, the Court again demurred with Gill v. Whitford, Benisek v. Lamone, and Rucho v. Common Cause; with an increasingly conservative bench, future action appears unlikely.

Historically and constitutionally, Federal districting reform is the purview of Congress. This paper aims to reinvigorate debate over the legislative remedy. The Constitution gives Congress powers over the “Times, Places and Manners of holding Elections.” (Article 1, § 4) Congress has exercised these powers to regulate the constituencies of the House of Representatives, and continues to do so. With the Apportionment Act of 1842, Congress required that states elect members from contiguous, single-member districts. (Cong. Globe, 1842) In 1872, Congress stipulated further that the districts be equipopulous. (17 Stat. 28, 1872) And from 1901 to 1929, districts were to be “composed of contiguous and compact territory and containing as nearly as practical an equal number of inhabitants.” (31 Stat. 733, 1901) In the immediate aftermath of the Reapportionment Revolution, both houses revived those requirements in 1967 only for the bill to fail in conference.¹ (Cong. Quarterly, 1968) Federal law today requires

¹The disagreement arose not over the requirements of compactness or contiguity, but over the level of population equality to be required before the 1970 Census, and the time allotted to states for redistricting in the interim. In the Senate version, the time allotted to the states was as low as three months. By 1967, the 1960 Census was quite dated. A new Census was impracticable, and there were concerns that strict rules and a tight clock would
only that representatives be elected from single-member districts. (US Code, Title 2, §2c, Amended 1967)

What would be the impact on minority and partisan representation in the House, of resurrecting the 1901 and 1967 requirements of contiguity, equipopulation, and compactness? This question directly confronts a long-standing conceptual and methodological hurdle. Though equipopulation and contiguity have precise, mathematical definitions, consensus over compactness has proven less forthcoming. It is not for want of trying. Literatures in law and political science (as well as computer vision, urban studies, and limnology) are awash with mathematical definitions of compactness, quantifying the closeness of people within a district, the length of its perimeters, or the similarity of its shape to a circle (see Section I or Angel et al. (2010); Chambers and Miller (2010); Ehrenburg (1892); Forrest (1964); Fryer and Holden (2011); Grofman (1985); Hofeller and Grofman (1990); Niemi et al. (1990); Polsby and Popper (1991); Reock (1961); Schwartzberg (1966); Young (1988)). Which definition ought to be applied? Does it matter?

To address this question, I compare statewide plans optimized for each definition of compactness. Even armed with a formal mathematical definition of the objective, this type of optimization problem is not exactly solvable with computers. The number of potential solutions is combinatoric in the number of inputs, and an exhaustive search is not feasible. To make progress, heuristic, iterative strategies may be employed. A search from a single initialization (seed), will terminate with a local extremum. Such solutions (maps) will be better than neighboring configurations, but are not guaranteed to converge to the spatial configuration or the value of the objective function, of the global optimum. Nevertheless, repeated initializations and searches allows for the assembly of a collection of high-quality maps for each objective. One can then calculate expected political outcomes – party or minority representation, for instance – for each map so generated. Doing this transforms the collection of maps into a distribution of practical outcomes for each objective function. That is the strategy adopted in this paper: armed with a procedure for generating maps for any objective, I evaluate and compare the practical implications of a choice among objectives, for the two parties and for minorities.

This approach builds on a long history in “automated districting,” that stretches from Weaver and Hess (1963) at the dawn of computation, to modern projects like Altman and McDonald (2011) who wrote software but did not exercise it, and Fryer and Holden (2011) who generated individual maps using “power diagrams.” More recently, Chen and Rodden (2013) and Cho and Liu (2016) deployed automated districting procedures to generate “populations” of districts, and measure if enacted plans were outliers with respect to these distributions. Fifield et al. (2017) criticize the statistical rigor of these methods by showing that compact districts generated by a single algorithm do not reproduce the distribution of political outcomes of the full population of contiguous maps. This finding is unsurprising in the sense that contiguous maps obviously represent a different distribution than ones optimized by particular

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2 This class of problem is called “NP-hard” (non-deterministic polynomial time, hard) in the computer science literature.

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force many states to elect members at large. This would throw the House into disarray, and it therefore rejected the Senate version. It is worth noting that the Court’s assumption of authority over Congressional districts in the Reapportionment Revolution has been a mixed blessing. It has privileged equipopulation above all other considerations, and it may have discouraged appropriate and necessary action by Congress. See Appendix E for a legal history.
algorithms. But at a deeper level, one might imagine that maps created by different compactness algorithms might also result in different distributions along political outcomes. This paper extends earlier work by implementing multiple definitions of compactness, and evaluating their practical impacts on the parties and on minorities over many states and elections. The software implemented is also able to generate larger numbers of plans for larger states than past projects. (Appendix B provides a technical review of existing work on automated districting and regionalization.)

Using this software, I generate statewide plans for many different compactness definitions, and several stand-alone algorithms. I demonstrate that the choice among definitions is not itself a loaded, political issue. Among the 10 states studied, the definitions treat the two parties consistently in terms of vote shares, seat shares, and the number of competitive races. I then use a single definition of compactness to district the entire United States. I evaluate the potential effects on minority representation and find them to be small. The work thus departs from past reviews by suggesting that compactness represents a meaningful, unbiased standard for legislative, Federal districting reform.

1. Compactness

Compactness is a succinct proxy for proximity-based communities. It reflects the American norm of geographic representation for the House that, though not mandated by the Constitution, was clearly expected by the founders and is now statutorily required under Article 1, §4 powers. This Section presents a number of common expressions for measuring compactness, which will be used as objective functions in the optimization procedure presented in Section 2.1. It describes some of their limitations, in that context. It also presents several stand-alone procedures or algorithms for generating compact districts.

1.1. Defining Compactness. In an early review of compactness, Niemi et al. (1990) remarked that it is “multidimensional.” As illustrated in Figure 1, the single word corresponds to many different notions. While a circle is broadly understood to be compact, shapes may be non-compact by being disperse (or distended), highly indented, or dissected. A single shape may be judged more or less compact based on the proximity of the elements it contains.

Each separate notion corresponds to a separate mathematical expression. Not all notions of compactness are created equal. In this work, I reject mathematical definitions that are not invariant under rotations or scaling. The compactness of a district does not change when measured in miles instead of kilometers or when surveyed with a compass instead of a sextant. This precludes concepts like the total perimeter of a district, or the ratio of its North-South and East-West extents. I further require that each measure of compactness be normalized to 1, and that a larger number is always more compact. This requirement makes compactness comparable across states and districts with very different population density, and makes it easier to combine it with other constraints, namely equipopulation.

Most measures of compactness are composed of ratios of lengths, areas, or populations of the district, with respect to a reference shape derived from the district. Figure 2 illustrates the common reference shapes. Four circles are defined: the largest inscribed circle (LIC), the smallest circumscribing circle (SCC), and the circles of equal area and equal perimeter. I define the radius of the equal area circle $R \equiv \sqrt{A/\pi}$, and call the circle of radius $R$ centered at the district’s centroid $C_R$. The convex hull (CH) is defined as the smallest convex polygon that encloses a district; it is the shape a rubber band would make if wrapped around it.
For each shape, I denote its surface (or shape) by $S$ and its area by $A$, the perimeter by $P$ and the length of that perimeter by $\ell$. The population contained within the shape is $p$. One may additionally define radii to the centers of population $\rho_p$ or area $\rho_A$, and distances to the perimeter $d_P$ or between two points $d_{ij}$. The shortest internal path $\delta_{ij}$ is the distance between two points, constrained to lie within the shape (see Figure 2). In what follows, I denote intersections by $\cap$ and averages by $\langle x \rangle$. I use subscripts to denote variants of these quantities, for the derived shapes.

With these definitions in hand, it is straightforward to define “classical” compactness measures from various ratios. The measures are tabulated in Table 1 and described below.

1.1.1. *Isoperimeter quotient.* Perhaps the most-famous measure of compactness is the isoperimeter quotient (IPQ); it is defined as the ratio of a shape’s area to that of a circle of equal perimeter. A circle of circumference $\ell$ has area $\pi (\ell/2\pi)^2 = \ell^2/4\pi$, so the IPQ simplifies to $4\pi A/\ell^2$. In the districting literature, it is often attributed to Polsby and Popper (1991), while IPQ$^{-1/2}$ is associated with Schwartzberg (1966). The IPQ is mainly sensitive to the perimeter; it responds little to broad deformations of the shape. It and other perimeter measures exhibit subtle definitional issues, because geographic boundaries like coastlines often have “fractal” properties in the sense that their length depends on the scale at which they are measured. Nevertheless, the method is computationally simple, readily understood, widely used, and fairly performant. I drop a number of monotonic transformations of the IPQ (like the Schwartzberg measure).

1.1.2. *Convex hull ratios.* The convex hull may be used to define numerous metrics by dividing the areas or populations in the district with those in the hull. To recognize the existing geometry of the state (whose borders may not be compact), the population or area of the hull may be limited to citizens or land within the same state. I implement the area ratio $A/A_{CH}$ and “population polygon” method $p/p_{CH,\text{state}}$. These typically privilege convex shapes and results in maps with clean, convex districts that may however be fairly disperse or “long.”

1.1.3. *Moments of Inertia.* The moment of inertia $I$ is a dispersion measure defined by the weighted distances squared of the elements of a district to a fixed point. Weaver and Hess implemented its application to the districting problem as early as 1963. In this case, the weights $w_i$ of the cells are their populations or areas, and the reference point considered is the center of mass (either area or population). For discrete elements $i$ on the surface $S$, this is $I = \sum_{i \in S} w_i \rho_i^2$.

A uniform circular disk of equal area is typically used as the reference shape. The moment of inertia of such a disk with respect to its center is $A \sum_i w_i/2\pi$. For areal weights this simplifies to $A^2/2\pi$ and for a disk of population $N$ it is $NA/2\pi$. Since large moments of inertia denote less compactness, the normalization is in the numerator: $2\pi (\sum_i w_i \rho_i^2) / (A \sum_i w_i)$.

1.1.4. *Inscribed and Circumscribing Circles.* In 1892, Ehrenburg proposed considering the ratio of a shape’s area to that of its LIC or SCC: $A_{LIC}/A$ and $A/A_{SCC}$. Reock proposed the latter measure again in 1961, and it sometimes bears his name in the districting literature. In practice I have found that they require heavy-handed optimization with ad hoc ‘fixes’ (see Appendix C.4.1). The problem is that any change to the shape that does not touch the circle is equivalent, so that there is no penalty for ‘tentacles’ and no ‘smooth path’ towards a global minimum.
1.1.5. **Exchange index.** Angel et al. (2010) propose to calculate “exchange” as the ratio of the areas of the intersection of an equal-area circle centered at the district’s centroid, with the district itself: \( A(S \cap C_R)/A \). The larger the fractional intersection, the more compact the shape. Because it privileges modifications to districts’ boundaries that place more of the area close to the center, the definition has a smooth “path” towards an optimal configuration and works well in automated settings. It is in some sense a dispersion measure.

1.1.6. **Mean, Dynamic, or Harmonic Radius.** The mean radius is the average value of the radius \( \rho \) to the district centroid. The dynamic and harmonic radii instead express \( \rho^2 \) and \( 1/\rho \), respectively. (Frolov, 1975) All three are areal dispersion measures. Integrating the radius \( \rho \) over a surface \( S \) of area \( A \), the mean radius is thus \( (\int_S \rho dS)/S \), the dynamical radius is \( \sqrt{\langle \int_S \rho^2 dS \rangle}/A \), and the harmonic radius is \( A/\int_S dS/\rho \). Each of these may be normalized by the corresponding value for a circle of equal area: \( 2R/3 \), \( R/\sqrt{2} \), or \( R/2 \), respectively. Since radii larger than a circle’s are less compact, the normalizations go in the numerator.

1.1.7. **Distance to Perimeter.** The average distance from a point in a shape to its perimeter \( d_P \) is compared to a circle, for which the value is \( R/3 \). More-compact districts have less of their area close to the perimeter.

1.1.8. **Path Fraction.** Chambers and Miller (2010) propose a measure of “bizarreness,” which reduces to the probability that the shortest in-state path between two people in the district, is itself contained within the district. The intuition is that a representative should not have to leave her district when driving from one voter to another. For people \( i \) and \( j \), this is \( \left( \sum_i \sum_j |d_{ij}/\delta_{ij}| \right)/N^2 \).

1.1.9. **Interpersonal Distance/Power Diagrams.** Fryer and Holden (2011) use the total distance squared between people. If the people are aggregated into cells (here, census tracts) \( i \) or \( j \) with populations \( w \) and separated by distance \( d_{ij} \), this is \( \sum_i \sum_j w_i w_j d_{ij}^2 \). Fryer and Holden demonstrate that this measure corresponds to an additively-weighted power diagram (like a Voronoi diagram) and, taking compact to mean “proximate,” prove that it is “optimally compact.” In Section 1.2.1, I describe an explicit algorithm for power diagrams, but a normalized measure may also be defined by dividing the average interpersonal distance by the corresponding value for a circle of equal area, \( 128R^2/45\pi \).

1.1.10. **Axis Ratio.** The simple width to length ratio \( W/L \) is not very sensitive as a compactness measure: depending on the precise definition, a spindly “X” may be as compact as a square. But it is in fact used (at least, by Iowa) and it is readily calculable. The width and length may themselves be defined in a number of ways; I calculate \( W/L \) as the ratio of the eigenvalues of the covariance matrix of the population point cloud in the projected geometry.

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\( ^3 \)This normalization can be derived by using the law of cosines to calculate the distance from a point with \( \theta = 0 \) to another arbitrary point with \( \theta \in \{0, \pi\} \), and integrating, taking care to weight the radius to get a uniform distribution on the disk: \( \left( \int_0^\pi \int_0^{\sqrt{1+2r_1 r_2 - 2 r_1 r_2 \cos \theta}} r_1 r_2 d\theta dr_1 dr_2 \right) R/ \left( \int_0^\pi \int_0^{\sqrt{1+2r_1 r_2 - 2 r_1 r_2 \cos \theta}} r_1 r_2 d\theta dr_1 dr_2 \right) = 128R^2/45\pi \).
1.1.11. **Visual Test.** The visual test (Young, 1988) – sometimes jokingly called interoculular (it hits you between the eyes – Grofman, 1991) or pornography test (“I know it when I see it,” (Polsby and Popper, 1991, cite Abner Mikva)) – is in fact a serious legal and diagnostic tool. Justice O’Connor wrote in *Shaw v. Reno* that “reapportionment is one area in which appearances do matter” (509 U.S. 630, 1993). Recently, Chou et al. (2012, 2014) and Kaufman et al. (2017) have elicited visual feedback on district plans from both experts and laypeople, to understand what people perceive as unfair.

Moreover, the visual test is an indispensable diagnostic for debugging and evaluating if the code for other metrics are “working.” In that sense, it represents my own inescapable bias for this project: it is the threshold where maps looked sensible.

It is worth acknowledging that there do exist definitions of compactness that are not included above. There is a limit to what is computationally feasible for automation. For example, Angel et al. (2010) define a “traversal index” by dividing the average length of internal paths between points on a district’s perimeter by the corresponding value for a circle. Computationally, this is simply too demanding. Evaluating this method would entail reevaluating the shortest paths for every potential move, which is computationally unfeasible. This project could well be extended by implementing new methods and variants; part of the aim of this research is to motivate a broad strategy for comparing formal of districting objectives.

1.2. **Procedures.** In addition to metrics – scalars that can be used as objective functions in an arbitrary optimization – a number of algorithms or procedures have been defined for generating a compact districting for a state.

1.2.1. **Power Diagrams.** The first of these algorithms is the power-diagram method discussed above (1.1.9). My implementation is similar to Fryer and Holden’s, and proceeds as follows:

(1) Regions \( r \) are defined by a center \( x_r \) and power \( \lambda_r \). The initial centers are chosen randomly and the initial powers are set to 0.

(2) Cells \( c \) located at \( x_c \) are assigned to region \( r \) by \( \text{argmin}_r (|x_r - x_c|^2 - \lambda_r^2) \).

(3) The region centers \( x_r \) move slowly towards the region centroids and the powers \( \lambda_r \) increase or decrease so as to equalize the regions’ populations.

Steps 2 and 3 repeat until a convergence threshold is reached.

1.2.2. **Split-Line Algorithm.** The split-line algorithm iteratively splits the state’s regions. Regions (districts) \( r \) of population \( p_r \) containing \( s_r > 1 \) seats, are split in two pieces of population \( p_r[s_r/2]/s_r \) and \( p_r[s_r/2]/s_r \), along the shortest possible line. This proceeds until each region has a single seat. The algorithm was first conceived by Forrest (1964), and a slightly different approach is laid out by Spann et al. (2007)

1.2.3. **Areal or Population Radii.** For comparison purposes, I have included a distance-based assignment approach, similar to Chen, Rodden, and Cottrell (2013; 2015; 2016). In short, I trade cells between districts to minimize the cell’s squared distance to the population or area centroid. My method differs from theirs in that I weight the squared distances by the radius squared of the equal-area circle (in other words, the area divided by \( \pi \)). This makes the algorithm scale-invariant. I also allow the algorithm to run long beyond population convergence, to obtain more-compact districts. This approach can be naturally subsumed in the “general objective function” approach used for the other compactness scores.
This section has presented various meanings of compactness, and reviewed existing mathematical definitions of the term. I now turn to deploying these definitions, to algorithmically generate compact districts.

2. Automated Districting: The C4 Software

After each Federal Census, the states are apportioned representation in the House of Representatives proportional with their populations. The states are then tasked with assigning these seats to equipopulous, single-member districts of contiguous area.

This Section presents this problem mathematically and describes the software implemented to address it computationally. This project differs from earlier catalogs of definitions and past efforts at automization in the diversity of objectives implemented. I am therefore able to quantitatively assess the practical impact of alternative definitions of compactness. Appendix B reviews past work on this problem.

2.1. The Constraint Problem. Formally, electoral districting is a graph partitioning problem. The task is to partition a set (state) into \( N \) non-overlapping, contiguous, equipopulous regions \( r \) (congressional districts), while optimizing the compactness of those regions. I do this using discrete cells \( c \) of population \( p^c \) (census tracts). Each cell is a node in the graph of the state, and the nodes are connected by edges if they are contiguous (share perimeter). The graph of the state itself must be connected: for any two nodes on the graph, there must exist a path between them. (See Appendix C.2 for details on islands and enclaves.)

The regions are then connected subgraphs of the state, and partition it: every cell in the state must belong to exactly one region. I denote the set of cells (nodes) in each region by \( X^r \). The regions’ populations are the sum of their cells’s populations, \( p^r = \sum_{c \in X^r} p^c \). The target population across regions \( p_{\text{target}} \) is equal to the population of the state, divided by \( N \).

The compactness of a region is a function of its cells, \( C_{p^r X^r} \). I will refer to the region that contains \( c \) by \( r(c) \).

The contiguity requirement is algorithmically enforced: no change that results in a disconnected graph for any region is ever considered. It is useful, however, to define the set of nodes \( X^{r'} \) that are not in \( X^r \) but are adjacent to a node in it, and are not themselves cut-nodes of their current region (their removal would not break its connectedness). Considering a cell \( c \) in \( X^{r'} \) and \( X^s \), I then define the union of \( X^r \) with one additional node \( c \) by \( X_{c}^{r} = X^{r} \cup \{c\} \), and the set with one node removed by \( X_{-c}^{s} = X^{s} \setminus \{c\} \).

The contiguity requirement is thus built in to the procedure. The equipopulation constraint and compactness objective are explicitly optimized using a greedy search that proceeds cyclically over the \( N \) regions.

Naïvely, one might define a combined objective function, incorporating the compactness and population count of each region. In each iteration a region \( r \) would annex the cell \( c \in X^{r'} \) whose reassignment from its current region resulted in the largest improvement in the combined objective function of the two regions. Along these lines, the population objective for each region might then take the form \( \mathcal{P}(p^r) = - (|p^r/p_{\text{target}} - 1|/\Delta)^{\alpha} \), with \( \Delta \) an allowable tolerance from \( p_{\text{target}} \), and \( \alpha \) a tunable parameter that I set to 4. The gradient of the population constraint would thus plummet as \( p^r \) approached within \( \Delta \) of \( p_{\text{target}} \) (since the parenthesis is less than 1, raised to the fourth), but it would dominate the spatial part when \( |p^r/p_{\text{target}} - 1| > \Delta \). One would then consider changes in this objective from moving a cell \( c \) from region \( r \) to \( s \): \( \mathcal{P}(p^r - p^c) + \mathcal{P}(p^s + p^c) \). This approach fails because the cells do not have equal population. Restricted to discrete trades, far from equilibrium, cells with larger population will always
move first. Roughly speaking, the step size is much longer among more-populous cells but may not lead in the direction of steepest descent.

A small modification of the above suffices, but comes at the price of an explicit objective function. I define the population difference function by

$$
P(p^r, p^s) = \text{sign} (p^r - p^s) \left( |p^r/p_{\text{target}} - p^s/p_{\text{target}}| / \Delta \right) ^\alpha . \quad (1)
$$

This expression depends only on regions, and is independent of cells. The population constraint thus impacts the choice of region to trade with, while the choice of cell along that border is left to the compactness scores $C$. As above, this term dominates the compactness measure when two regions’ population difference exceeds $\Delta p_{\text{target}}$ but is very small when equipopulation is satisfied.

Each iteration on a region $r$ culminates by its annexing the cell that maximizes the combined population and compactness function:

$$
\arg \max_{c \in X^r} \left[ P(p^r, p^{r(c)}) + C(X^r_{+c}) - C(X^r) + C\left(X^{r(c)}_{-c}\right) - C\left(X^{r(c)}\right) \right]. \quad (2)
$$

The optimization procedure begins by seeding the $N$ regions with $N$ random cells, and the regions initially grow by subsuming unassigned cells $u$. Since the unassigned area has target $p_{\text{target}}^u = 0$, the population score to transfer out of it is infinite. In practice, I replace this score with a large number so that the arg max is well-defined, but the behavior is unaltered: the regions quickly converge to cover the state. The procedure thus partitions the state while respecting the contiguity of regions, and optimizing for equipopulation and compactness.

I also implement a modification of this procedure. In addition to one-directional moves, it is efficient to be able to trade cells between regions. If this functionality is activated, one trade is allowed per cycle, for which $c \in X^r$ and $c' \in (X^{r(c')} \cap X^r)$ yield the largest gain in compactness:

$$
\arg \max_{c,c'} \left[ C(X^r_{+c,-c'}) - C(X^r) + C\left(X^{r(c')}_{-c,-c'}\right) - C\left(X^{r(c')}\right) \right]. \quad (3)
$$

2.2. Computational Implementation. A core contribution of this project is the software used to generate optimized districting plans. The software is called $C_4$, for “contiguity-constrained clustering in c++.” $C_4$ is open-sourced and freely available on GitHub. Key features are presented in greater detail in Appendix C.

To begin, a user loads the cells for a state along with their adjacency matrix (shared perimeters). To be able to enforce contiguity, the statewide plan must initially be connected; islands’ connections to the mainland may be specified explicitly, but $C_4$ also has a module to handle this automatically. $C_4$ also subsumes regions that are connected to the main graph by a single cut vertex, since it is definitionally impossible to reassign the cut vertex without breaking contiguity.

The search then begins with a random draw without replacement of one cell for each region. The hill-climbing procedure detailed above then begins. Critical to the algorithm’s performance is its enforcement of region contiguity using integer programming. This is done by requiring that any cell removed from a region leave its neighbors in a single, connected subgraph. The search terminates after a configurable number of cycles with no improvement.

The software includes several of the standard metaheuristic strategies; tabu lists (Glover, 1989) are in fact used for some compactness objectives, with individual cells precluded from moving for a fixed number of iterations after reassignment. I further implement two non-standard procedures. First is a “de-strand-ing” method that removes strands of cells that
cannot be removed by the cell-by-cell search. Second is a method for re-starting the search, by splitting in two the region with the worst compactness score and merging two other regions.

3. The Political Consistency of Compactness Definitions

This Section presents the core analysis, contrasting the spatial and political characteristics of maps generated with the $C_4$ software with those from enacted plans. For this work, the fundamental cell size is the Census tract, and equipopulation is required at the 2% level (with a few exceptions, below). Readers may object that Federal law allows districting at the Census block level and that the Supreme Court has rejected any *de minimis* threshold of equipopulation. These choices have the obvious advantage of speeding up the computation, though most of the algorithms work fine at the block group level. But they should also be considered in the context of the legislative approach that motivates this paper. Census tracts are designed to encapsulate relatively-homogeneous populations and their use can be thought of as a minimal regard for “preservation of communities.” When the Congress last considered legislation to require equipopulous districts in 1967, it was at the 10% level – a looser threshold in better balance with other districting objectives. (Cong. Quarterly, 1968) As to the Court’s enforcement of its “one-person one-vote” doctrine, it would be faced with Congress’s explicit Article 1, Section 5 authority over the elections and qualifications of its members.

I have generated over a thousand maps per measure for each state for which I have voting data, using distributed computing. The exceptions are the path fraction, where I have generated only 280 maps per state, and the split-line algorithm, which is deterministic and yields a unique solution per state. The optimization procedures occasionally fail to converge; the following analysis is therefore restricted to those maps with population deviations less than 2%. The population convergence issue is more acute with the axis ratio method and for Texas, and in these cases I allow a 5% deviation; for axis ratio maps of Texas, a 10% threshold is allowed. The split-line algorithm generates a 2.1% deviation in Illinois and that solution is retained.

One must be precise about the statistical nature of this collections of maps. The algorithms are initialized by selecting one cell (Census tract) to seed each district, without replacement. This is a bona fide random draw. The optimized districts are of course not random.

After some general observations about the visual consistency of methods, I analyze the consistency of the compactness measures in two ways. First, I study the political outcomes of the populations of “optimized” maps: the seat share and competitiveness. I then turn briefly to the potential impacts for minority representation.

3.1. Observations on Optimized Maps. In Figure 3, I present a representative collection of maps drawn from a single seed. Additional maps can be explored online through an interactive map. Differences between methods emerge as expected. Axis ratio is simply ineffective. Isoperimeter quotient contains “somewhat lumpy” shapes with smooth perimeters. The hull-based measures, along with the power diagram and split-line algorithms, produce convex shapes with straight lines. It is interesting to consider the non-trivial relationships between the many methods. Power diagrams imply convex shapes that would result in good scores for hull population or hull area, but the converse is not necessarily true: convex shapes can be very distended (disperse) while power diagrams usually are not. A convex shape will contain all of the paths between people in the district and will therefore have a “Path Fraction” of 1 (§1.1.8), and a shape with a perfect “Path Fraction” likewise implies a high convex hull
population ratio (§1.1.2), but the paths through phase space towards these optima are not generally the same.

Across measures, the varying treatment of Pittsburgh is particularly notable: some algorithms divide it in many pieces (distance to the areal center or split-line) while others cut a circle around the city (exchange, harmonic radius, or inscribed circles). It is this variation in the treatment of urban (in America, Democratic) voters, that raises the possibility of bias from compactness. Is choosing an objective equivalent to choosing a winner?

3.2. The Political Consistency of Optimized Maps. The seat share and competitiveness of simulated districts are derived by reaggregating precinct-level voting data from presidential elections, described in Appendix D. Presidential elections are used to avoid incumbency effects and uncontested races. The procedure depends on consistency between presidential and congressional races. It is also an approximation in the sense that local candidates could better tack to individual constituencies, and even change their strategies as a function of the district lines. Each individual map results in a certain number of projected wins for Republicans and a complementary number for Democrats; each measure’s population of maps thus corresponds to a distribution of seats for each election. The same procedure is followed for the actual enacted maps from the last three districting cycles. In this way, the internal consistency of the simulated maps may be evaluated, and as a group contrasted with the enacted maps. Results are shown for four elections in Pennsylvania in Table 2. The other nine states – Florida, Illinois, Louisiana, Maryland, Minnesota, North Carolina, Tennessee, Texas, and Wisconsin – are available in the Appendix. In Table 3, I tabulate the expected number of “competitive” seats with margins of victory less than 5% and plot the distribution of vote shares for each state and method.

Three major themes stand out from these results. The first theme is the remarkable consistency of the seat shares among the eighteen algorithms and metrics shown. This pattern is reproduced for all of the states studied. This result suggests that from the perspective of the seat share, the choice of the definition of compactness is immaterial. A similar, but weaker result emerges from the competitiveness of the seats in Table 3. Though the agreement is not quite as tight as for the seat shares, the various metrics put fairly consistent numbers of seats in play.

The second observation is that although Democrats won each of the four elections shown in Table 2 by at least 2.5%, they capture a majority of the eighteen seats only in the 2008 election, which Barack Obama won by more than 10%. This thus reproduces the earlier results on “unintentional gerrymandering” by Chen and Rodden: Pennsylvania Republicans enjoy a structural advantage from their demography, independent of any machinations by the State Legislature. This is due to Democrats’ “inefficient” clustering in Philadelphia and Pittsburgh. The same effect is also visible in the vote shares of Table 3, in particular for Illinois. Chicago voters are overwhelmingly liberal, and Democratic candidates can expect margins of victory that are “inefficient” for the party as a whole. This is apparent in the heavy left-hand tails. As in Pennsylvania, the effect is accentuated by the fact that the major metropolis is in the corner of the state: it is hard to dole these voters out to swing districts. Maryland has a different story. Democrats again have a substantial majority, receiving around 60% of the vote in presidential elections. Naïvely, this might be sufficient for the entire state to “go blue.” But Maryland Republicans are protected by their geography: they are concentrated in the panhandle and Eastern Shore which, for any reasonably compact partitioning, are sliced off as
two safe Republican districts. The take-away is the unsurprising fact that each state’s political geography affects the representation for the two parties.

Still, returning to Pennsylvania and comparing the expectations from the simulated maps to that of the map enacted for the 114th Congress, it is readily apparent that the Pennsylvania Republicans enjoy an additional one to two seat advantage through their control of the districting process. This advantage persists over several elections, and in 2000 and 2012 the expectation of 6 seats for Democrats is completely outside the distribution of seats simulated using any compactness method. Similar pictures emerge in Maryland and North Carolina, where the Democratic and Republican majorities enacted plans that yield seat shares outside the distribution from simulations. The last observation is thus that the simulation provides a baseline “unbiased” level, from which the observed deviations on enacted maps evidence intentional gerrymandering. Crucially, this conclusion is extremely robust to the method employed to generate the counterfactual.

3.3. Impacts on Minority Representation. Before advocating an automated, compactness-based approach, it is necessary to consider its effects on minority representation. Minority voting rights are constitutionally and statutorily protected, under the 15th Amendment and the Voting Rights Act. Though the VRA is somewhat cumbersome, it is effective: minorities in the US are represented at rates far closer to proportionality than in peers like Germany, France, and the United Kingdom that lack explicit legal frameworks for ensuring their representation. To study this, I have generated compact districts for the entire country using the power diagram algorithm. As noted, power diagrams are equivalent to optimizing on the interpersonal distance. A Principle Component Analysis of the compactness of historical Congressional districts from the last three districting rounds, shows that it is correlated to the first component of the PCA at 95% (Appendix G). They are also extremely fast to generate.

Armed with a population of maps, I aggregate the ethnic and racial composition of census tracts to calculate the Black and Hispanic fraction of each simulated district as I had previously done for the precinct-level votes. In Figure 4, I present the number of seats (actual and simulated) where the Black or Hispanic share of the voting age population (VAP) exceeds a given threshold. This exercise is predicated on the notion that fraction of a district’s VAP belonging to a racial or ethnic majority is the key determinant in its electing a minority representative. The vertical distance between the dashed and solid lines gives a flavor for the change in minority representation from moving from the status quo to compactness-based districting, if a single threshold triggered minority representation. The lines intersect in both panels. This suggests that if a high minority share were required to elect a minority representative, the currently enacted plans would result in higher minority representation than the power diagram maps. Conversely, if a low minority share were required, the power diagram maps could result in higher minority representation. Unlike the party share, where 50% is clearly the relevant threshold, there is no axiomatic level of minority presence for a minority candidate to be elected.

Acknowledging the complexity involved in measuring such a threshold, and recognizing that using a single value country-wide is a gross simplification, I offer two simple approaches. The first is to identify the value of the VAP fraction \( f \), such that the number of constituencies with minority share greater than or equal to \( f \) is matched by the number of minority representatives

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4Among developed nations, New Zealand’s dedicated seats for Maoria have been more successful. Canada, also has fairly high rates of minority representation in the lower house of its parliament. (Chowdry, 2015)
actually elected in the 115th Congress. At this level, each district with a larger minority share that does not elect a minority representative is compensated by another district with a lower share that does elect one. This is illustrated by a dotted horizontal line in Figure 4. There are 40 Hispanic and 46 Black representatives in the 115th Congress,\(^5\) which translates into fractions of 29% for Blacks and 40% for Hispanics.

Alternatively, Cameron, Epstein, and O’Halloran (1996; 1999) famously evaluated a probit model with the minority share of the VAP as the independent variable and an indicator for a minority representative as the dependent variable. Taking the simplest possible model with minority share \(x\), minority representation \(r\), intercept \(\alpha\), and slope \(\beta\), I fit the normal ogive \(r = \Phi(\beta x + \alpha)\). This simple approach ignores the interplay between Black and Hispanic populations and the role of primaries in determining the (Democratic) candidate (Lublin, 1999; Lublin et al., 2009), and it is also markedly coarser than the local, Ecological Inference approach usually adopted in litigation. But the intent here is also very different: to evaluate the impact of a national change for which voting data are unavailable. In the 115th Congress, the 50% crossing point for this model is 36% for Blacks and 51% for Hispanics.

Intuitively, these thresholds correspond to a sizable majority of the majority party. The thresholds are higher for Hispanic districts, due to higher eligibility and turnout among Blacks than Hispanics. Using the first approach to the threshold, the enacted and compactness-based maps produce an almost equal number of minority seats in the region of interest. To interpret the probit models, one must take the sum over districts of the probabilities of electing a minority. Doing this for the actual districts yields 46.2 Black and 39.9 Hispanic representatives, compared to the true values of 46 and 40. The same sum of probabilities with the simulated maps yields 42.1 Black and 38.0 Hispanic representatives. According to the probit, a pure power-diagram approach would then lead to a 9% reduction in Black representation, and a 5% reduction in Hispanic representation.

In practice however, VRA compliance means that districts constructed with a high minority share also typically have a partisan composition favorable to minority candidates. The power diagram generation does not fine-tune this correlation, and power diagram districts may therefore require a higher raw minority VAP fraction to elect a minority candidate. This caveat implies that the estimates of minority representation under power diagram districts are likely inflated.

This said, in the past several Congresses, growth in minority representation has outpaced growth in the minority share of the population. This suggests that the “threshold” for minority representatives is falling. This point is reinforced by earlier work by Grofman et al. (2001). To the left of the intersections between curves of the enacted and simulated maps in Figure 4, the simulated maps yield higher minority fractions. If the threshold continues to fall, the minority representation under a compactness-based approach may exceed that from the current patchwork of judgment-based law. Moreover, following the Supreme Court’s rejection of the VRA’s preclearance formula in *Shelby County v. Holder*, this may be an effective, preventative approach at the national level.

Further, it is worth noting that the machinery already described provides the means for sidestepping the potential reductions in minority representation apparent in this analysis. One

\(^5\)I consider the union of entries from the U.S. House of Representatives, Office of the Historian (2017a,b) and the U.S. House of Representatives, Press Gallery (2017). The delegates from the District of Columbia, the Marianna Islands, Puerto Rico, and the US Virgin Islands are all minorities, but I do not include them in this count. In California’s 34th district, Jimmy Gomez replaced Xavier Becerra in a special election. They are both Hispanic, and I count the district once.
could preferentially select maps with better minority prospects, from the sets of automatically-generated compactness-based districts. This approach is obviously extensible to any “secondary criteria” that one might wish to apply (competitiveness, etc.), but these more-normative considerations are outside the scope of this work.

4. Further Steps and Conclusions

This paper has presented a credible automated districting procedure that implements many compactness definitions and districting algorithms from the previous literature. Past compendia of compactness measures have not systematically implemented the proposed definitions in automated procedures, and have therefore not contrasted the implications of the proposals. Using this software, I have generated populations of maps for each measure and algorithm, for a number of states. Aggregating votes from presidential elections into the simulated districts, I have projected the “winner” of each district in each election. I have thus transformed the populations of maps into distributions of vote shares for seats, and seat shares for states. The party vote shares across seats and elections reflect the political geography of the states; their distributions are remarkably consistent across methods, for each state. In particular, there is good agreement in the integrals of vote shares above and below 0.5 (seat shares for the two parties), as well as between 0.475 and 0.525 (number of competitive seats). Using power diagrams to simulate hundreds of maps for every state in the country, I find that a purely compactness-based approach would result in small but noticeable reductions in minority representation.

The present work thus suggests a new strategy for evaluating the impacts of formal objectives for legislative districts, in the context of Congressional action on gerrymandering in the United States. It could be extended by incorporating alternative initialization strategies including graph theoretic approaches and hierarchical partitioning, or adding algorithms and compactness definitions. Studies like those of Chou et al. (2014) and Kaufman et al. (2017), that elicit human feedback on which measures yield the most appealing solutions could also be informative. Given the interest in protecting communities of interest and political subdivisions, it is worth formalizing and implementing objective functions to encode these adjacencies, and other “normative” goals (minority representation, competitive districts, etc.). Such measures could then supplement the population and spatial terms in the objective function.

Since the balance of this document has suggested that the various definitions are similar in practice, it is natural to ask which one to use. And after selecting a measure, how compact should the districts in a map be? These are questions without hard quantitative answers. Power diagrams converge quickly and reliably; they result in clean, convex polyons – which is generally desirable but sometimes results in split cities (see Figure 3). I show in an Appendix that a principle component analysis of the compactness of historic districts yields a first component that is strongly correlated with the interpersonal distance measure (which Fryer and Holden (2011) show to be minimized by power diagrams). If one were to adopt one of the other objectives, one could simply require the “maximum” level of compactness – convergence. Critics might protest that this strategy would imperil other traditional principles, like respect for subdivisions or communities of interest. This could be mitigated by defining inviolate cells or neighborhoods for districting as suggested above with Census tracts. As described in the context of minority representation, one could also select plans from the distribution that satisfy some other objective. Outright maximization is, however, the stance
the Supreme Court adopted on the question of equipopulation in the *Baker* series (with *Kirkpatrick* and *Karcher*): it rejected any *de minimis* level, and this “bright line” standard has had an important preventative effect.

Of course, a new Apportionment Act is hardly the only proposed solution to political gerrymandering, nor is it the only one hinging on a clearer definition of compactness. Since the Court’s tentative endorsement of global measures of partisan symmetry in *LULAC*, Grofman and King (2007) and others have evaluated its use or like Stephanopoulos and McGhee (2015) have proposed improved versions, like the “Efficiency Gap.” But Justice Robert’s famously dismissed this as “gobbledygook.”

Conversely, compactness is consistent with the American tradition of geographic representation, and the Court has stood by it over time as a “traditional districting principle.” Notwithstanding, compactness has its critics. Grofman (1985) has suggested that it is almost accidentally relevant, “a useful criterion only to the extent that it happens to coincide with other features...which are of value in themselves.” Stephanopoulos has advocated a rich conception of “geographic communities,” taking into account “transportation links between areas, membership in regional organizations, the level of residents’ commercial interaction, and their degree of reliance on the same media outlets.” (Stephanopoulos, 2012, p. 1433, 1396-1399) But the justices have made clear their preference for a “precise rationale.” The Court’s clear “one person, one vote” standard (*Gray v. Sanders*, 372 U.S. 368, 1963) led to a rapid elimination of malapportioned districts; it had a strong preventative effect. By contrast, the justices’ willingness to exercise their “gut” judgment to provide relief for racial gerrymandering has curtailed neither legislators’ voracity nor the ensuing litigation. North Carolina’s twelfth district has reached the Supreme Court on four occasions since 1990: *Shaw v. Reno*, *Shaw v. Hunt*, *Hunt v. Cromartie*, and *Cooper v. Harris*. Elsewhere, Stephanopoulos (2013) cites the harms caused by excessive litigation from legislators’ oversight of district lines. It is a harm poorly remedied by a rich measure. Simple and strict enforcement of compactness has a longer history, and would likely be more effective.

Finally, in most of the developed world, redistricting is performed by independent commissions. These commissions must be charged with their objectives; should compactness rank among them, a clear definition is imperative. The present work has suggested that a choice among definitions need not be politics in disguise.

Automated generation of compact districts is not the only solution to gerrymandering, and it is perhaps not a complete one. I contend however, that it is likely to be a useful tool for any solution – legislative or otherwise. A quantitative understanding of the implications of formal districting objectives is critical to both research and reform.

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Table 1. Metrics of compactness, with formulae. See Section 1.1 for notation.
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Table 2. Votes from presidential elections in Pennsylvania are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 18 assigned after the 2010 Census.
The vote shares accruing to Republicans are plotted for all districts of each map, and for all available elections, leading to one distribution for each state and method. The consistency in the shapes of the distributions across methods suggests that the many methods do not differ in their treatment of the two parties. The different shapes for the four states shows the impact of political geography on partisan representation. Republican vote shares in excess of 0.5 correspond to Republican wins; the integral up to 0.5 corresponds to the Democratic seat share, as shown for Pennsylvania in Figure 2. The part of the distribution close to 0.5 are competitive races. To the left of each distribution, I tabulate the number of competitive races, calculated as the integral of the vote share distribution between 0.475 and 0.525. As for seat shares, the level of competitiveness is quite consistent across measures.

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</table>

Table 3. The vote shares accruing to Republicans are plotted for all districts of each map, and for all available elections, leading to one distribution for each state and method. The consistency in the shapes of the distributions across methods suggests that the many methods do not differ in their treatment of the two parties. The different shapes for the four states shows the impact of political geography on partisan representation. Republican vote shares in excess of 0.5 correspond to Republican wins; the integral up to 0.5 corresponds to the Democratic seat share, as shown for Pennsylvania in Figure 2. The part of the distribution close to 0.5 are competitive races. To the left of each distribution, I tabulate the number of competitive races, calculated as the integral of the vote share distribution between 0.475 and 0.525. As for seat shares, the level of competitiveness is quite consistent across measures.
Figure 1. Various distinct “concepts” are associated with compactness. The circle is typically agreed to be the most compact shape, to which may be contrasted disperse, indented or dissected ones. But one may also look “within” a shape. In the middle frame, the dots are less disperse if one shifts the cells from the thick black bounds to the dotted ones.

Figure 2. Building blocks of compactness measures, on Pennsylvania’s 7th congressional district. From these derived shapes and lengths, a great number of measures may be defined.
Figure 3. Representative districting plans of Pennsylvania, for various metrics. The treatment of Pittsburgh, halfway down in the western part of the state, evidences how optimizing according to different definitions of compactness results in different treatment of cities.
Figure 4. Presented are the number of districts whose population exceeds the shown thresholds of Black or Hispanic voting age population (VAP). For example, there are 73 districts whose population is at least 20% Black, and 46 districts whose population is at least 30% Black. The composition of the 115th Congress, with 46 Black and 40 Hispanic lawmakers, is represented by the dotted horizontal lines.
### Appendix A. Expected Seat Shares for Additional States

<table>
<thead>
<tr>
<th></th>
<th>2008 Presidential</th>
</tr>
</thead>
<tbody>
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<td>111th Congress</td>
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<tr>
<td>114th Congress</td>
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<tr>
<td>Axis Ratio</td>
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<tr>
<td>Circumscribing Circles</td>
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<td>Distance to Perimeter</td>
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<tr>
<td>Dynamic Radius</td>
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<td>Hull Area</td>
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<td>Hull Population</td>
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<td>Isoperimeter Quotient</td>
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</tr>
<tr>
<td>Split-Line</td>
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</tr>
</tbody>
</table>

**Table 4.** Votes from presidential elections in Florida are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 27 assigned after the 2010 Census.
Table 5. Votes from presidential elections in Illinois are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 18 assigned after the 2010 Census.
Table 6. Votes from presidential elections in Louisiana are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 6 assigned after the 2010 Census.
Table 7. Votes from presidential elections in Maryland are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 8 assigned after the 2010 Census.
Table 8. Votes from presidential elections in Minnesota are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 8 assigned after the 2010 Census.

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<td>114th Congress</td>
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Table 9. Votes from presidential elections in North Carolina are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 13 assigned after the 2010 Census.
### Table 10

Votes from presidential elections in Tennessee are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 9 assigned after the 2010 Census.

<table>
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<tbody>
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<tr>
<td><strong>Harmonic Radius</strong></td>
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<td><strong>Mean Radius</strong></td>
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<td><strong>Path Fraction</strong></td>
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<td><strong>Population Distance</strong></td>
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<td><strong>Power Diagram</strong></td>
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<tr>
<td><strong>Split-Line</strong></td>
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Table 10. Votes from presidential elections in Tennessee are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 9 assigned after the 2010 Census.
Table 11. Votes from presidential elections in Texas are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 36 assigned after the 2010 Census.
## Table 12

Votes from presidential elections in Wisconsin are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 8 assigned after the 2010 Census.
APPENDIX B. AUTOMATED DISTRICTING PROCEDURES

Automation has long held promise as a “solution” to gerrymandering. As early as 1961, Vickrey wrote that “elimination of gerrymandering would seem to require the establishment of an automatic and impersonal procedure for carrying out a redistricting.” By just 1963, Weaver and Hess began applying infrastructure built for the “warehouse problem” to districting in small cases (Delaware).

In the decades since, political observers presumed that with the political will, the districting problem would wilt before the burgeoning capacity of computing. In *Karcher v. Daggett* for instance, Brennan opined optimistically that “rapid advances in computer technology [...] make it relatively simple to draw contiguous districts of equal population and at the same time to further whatever secondary goals the State has.” (462 U.S. 725, 1983) This has taken longer than expected. Districting is exactly the type of (NP-hard) categorization problem that computers struggle with. So while computers quickly proved useful for tabulation and error checks, humans have until very recently performed much of the partitioning by hand. (Altman et al., 2005) Yet with appropriate metaheuristics and approximations, solutions are now in reach. In the past few years, a number of scholars have implemented serviceable automated procedures for individual metrics or algorithms. (Chen and Rodden, 2013, 2015; Cho and Liu, 2016; Chou et al., 2012; Fryer and Holden, 2011; Spann et al., 2007; Li et al., 2014)

It is critical to recognize that none of the software here terminates with an exact solution — the optimal result — and that the software proposed in Section 2.2 does not either. In other words, there is no guarantee that the partition generated corresponds to the highest possible value of the objective function. Set in the context of the partitioning of Pennsylvania discussed earlier, none of the methods here consider every one of the $10^{3800}$ possibilities. That is the nature of NP-hard problems. Instead, the algorithms begin are seeded with an initial configuration and iteratively move towards a “better” solutions. By initializing the algorithm many times with different starting points, it produces a variety of different “good” outputs — what I will call a “population” of maps. Fifield et al. (2017) criticize this approach and have sought to situate it on firmer theoretical foundations using a Markov Chain Monte Carlo algorithm. But they demonstrate full-state simulation only for New Hampshire, and generate local modifications to the districting plan in Pennsylvania. To be clear about the statistical meaning of this project, the initialization (Section C.1 is a bona fide random draw of points (Census tracts) in the state; the final output of the optimization is of course not random. Depending on the stability of the minima, a few maps may be represented many times.

Altman and McDonald published the first modern salvo at a comprehensive “solution” to automated districting in 2011, with BARD (Better Automated ReDistricting). They implemented several objective functions and metaheuristic search procedures but did not demonstrate the results from the program and claimed only to have had “moderate success with moderate size plans.” The package has not been maintained, but Liu et al. (2016) did attempt to use it and found it slow. The issues seem to have been the choice of language ($R$), inefficient objective functions, and the use of a contiguity score instead of a constraint.

The same year that Altman and McDonald produced BARD, Fryer and Holden (2011) proposed to evaluate non-compactness via a “relative proximity index” (RPI). They defined the RPI as the average distance between people in a state’s districts, divided by the average distance for an optimal plan. Assuming that an “optimal plan” would (i) respect anonymity,
(ii) minimize distances between voters and (iii) be scale invariant, they prove that it is a power


diagram and implement an algorithm to generate solutions (see Section 1.2.1).

In the last several years, Chen and Rodden (2013; 2015) and Cho and Liu (2016) have
proposed to measure gerrymandering by comparing a population of “potential plans” to en-
acted ones. These too have required functional algorithms. Chen and Rodden implemented
a fairly straightforward algorithm, beginning from random seeds, merging cells to form an
(agglomerative) initial solution, and then trading neighbors based on proximity until the pop-
ulation equality is satisfied. Their initial analysis focussed on Florida, but they did apply the
algorithm to twenty states, and Chen and Cottrell subsequently applied it to the full country.
(2016)

Cho and Liu implemented a genetic algorithm, but allow that their crossover mechanism
is “sufficiently disruptive” that it is used infrequently, and may not be much better than re-
initialization. It is thus effectively a GRASP algorithm: greedy with random mutations. What
distinguishes their work is the care given to the preservation of contiguity and the diversity of
mutations that they allow. In short, contiguity is built-in – it cannot be violated. Given a set
of cells to be removed from a region, their algorithm checks that the set’s external neighbors
in the source region are themselves connected. This purely-local check only works however,
because of their choices to preclude holes in their plans and use queen instead of rook weights
for contiguity.7 Though they discuss the extensibility of their framework, they implement
only the IPQ measure, which is computationally extremely simple. They use the National
Center for Supercomputing Applications to generate millions of solutions, which is impressive
but statistically unnecessary and hampers reproducibility. They only demonstrate plans for
Maryland and do not give a good sense of the overall quality of their plans.

Kimbrough et al. (2011) and Chou et al. (2012, 2014) also implement a genetic algorithm
with no crossover, to optimize an extremely simple measure of compactness: the maximum
intradistrict distance. The crux of their series of papers is a lovely suggestion to use Interactive
Evolutionary Computation (IEC) with what they call solution pluralism. In other words: they
ask actual people to compare plans, and use these to validate the performance of the IPQ,
moment of inertia, and perimeter measures. They do this with subjects both in-person and
through Amazon Mechanical Turk. This work would then ideally lead to a collection of
solutions for debate.

Of course, the regionalization literature extends far beyond districting. Duque et al. (2011)
provides an outstanding review of regionalization methods, with a useful classification of
heuristics. Though this project focusses on trading optimizations, the graph theoretic and
hierarchical/agglomerative strategies he reviews could usefully extend this work. From a prac-
tical perspective, Li et al. (2013, 2014) propose a generic tool for compact regionalization with
goals similar to the present work. Their tool optimizes a normalized moment of inertia, and
they compare the performance of greedy, GRASP, and tabu searches. They obtain their best
results from the randomization in GRASP.

Finally, a number of regionalization algorithms are available in clusterPy (Duque et al.,
2011). Unfortunately, the feature set is not well-suited to the districting problem. First, the
objective function is fixed (maximum in-region homogeneity on intensive variables). Further,
though many of the included algorithms allow the user to specify the number of regions, they
do not provide the possibility of enforcing a minimum population deviation among the regions.

7With “queen” contiguity, elements touching at a single point are considered contiguous, whereas “rook”
quantity requires the contiguous elements to share non-zero perimeter. Cho et al state that their choice of
Queen weights and no holes is legal rather than computational, but do not provide sources for this.
The notable exception is the max-$p$ algorithm (Duque et al., 2012), which is also available in PySAL. (Rey and Anselin, 2010) Though max-$p$ generates the number of regions endogenously, one can easily obtain the desired number by requiring each district’s population to match the target. The algorithm does not intrinsically privilege geographical compactness, but this can be achieved by providing latitude and longitude as clustering variables.

**Appendix C. The C4 Software**

*C4* is a multi-objective application for contiguity-constrained clustering. It is implemented in C++, with bindings to Python with Cython. This allows it to draw on the strengths of the two languages: C++ for speed and Python for easy configurability and post-processing (mapping, analysis, etc.). I use PostGIS/Postgresql geographic database software to prepare the inputs in two ways: I simplify the boundaries between census tracts, and I extract the contiguity matrix among the cells in a state. This pre-processing stage can be cached, so there is no dependence on a private database. The software is open-sourced and freely available on GitHub.

The core code is implemented as three classes: (1) a universe which owns a collection of (2) regions and (3) cells. In the present application, regions are congressional districts and cells are census tracts. Speed requires explicit memory management: the cells are instantiated once by the universe and never copied. Each region maintains a vectors of pointers to the cells currently assigned to it, and additional vectors for the cells along its internal and external borders. The cells hold their properties (location, area, perimeter, population, etc.) as well as pointers to their neighbors.

To start, the user loads the cells and adjacencies, and starts an initialization round that creates a contiguous solution from which to iterate. The iterative search then begins, trading cells between regions to equalize populations and improve one of the compactness metrics described above. Upon convergence, the program returns a list of assignments of cells (census tracts) to regions (congressional districts).

C.1. **Initialization strategies.** A number of initialization strategies are implemented. The split-line and power diagram algorithms discussed earlier are stand-alone, and can be used to initialize a statewide plan. A simple $k$-means clustering approach is also provided, as well as with random contiguous growth. An existing plan can also be loaded.

However, for the studies presented in this paper, initialization consists simply of a random draw of a cell (census tract) for each region. This ensures that the single strategy under study is completely responsible for the outcomes presented.

C.2. **Contiguity.** Implementing contiguity as a constraint instead of an objective significantly improves performance. To do this, the contiguity graph of the state is extracted from its topology in Postgres. In this graph, each cell (census tract) is a node, and edges are drawn between adjacent cells (i.e., neighbors). To preserve contiguity in the search, the graph must be modified in two ways:

1. The statewide graph must be connected in the first place. *C4* provides an automated procedure that connects islands to their closest neighbors until the graph is connected, but it also allows the user to specify explicit connections. In practice, I connect islands and disconnected subgraphs to land based on bridges, ferry routes, and shared jurisdictions. For example, the Eastern Shore of Virginia is connected to Virginia Beach by the Bay Bridge, and San Nicolas Island, California is connected to Naval Base Ventura County, of which it is a detachment.
Figure 5. Census tracts of Northumberland County, Pennsylvania are shown with enclaves highlighted. For a trading algorithm to work consistently with a contiguity requirement, enclaves and islands must be assigned to their external neighbor or nearest land neighbor.

(2) “Enclave” census tracts must be assigned to their external neighbor. An example of this is shown in Figure 5. Without merging the highlighted cells, it would be impossible to move their outer neighbors without violating contiguity. This argument also applies to coastal tracts that are not enclaves of another tract, but whose entire land border is shared with a single tract. The topology is different, but the graph structure is the same: they are across a “cut vertex” from the main body of the district. It is worth noting that this procedure modifies slightly the definitions described above: the compactness scores and optimizations are evaluated on this merged (or “pruned”) topology.

After these modifications, any move can be required to preserve the source region’s contiguity check: a cell’s neighbors in the source region must be members of a connected subgraph of the source region. This check resembles Cho and Liu’s local check, but it differs in that the neighbors do not have to be directly connected. If they are not, the search extends through the neighbors’ neighbors until it is shown that the removal of the cell would result in disconnected subgraphs in the source region. The result of this distinction is that both rook and queen weights are allowed, as are holes. In practice, I have used rook contiguity.

C.3. Hill Climbing. Each iteration of $C^4$ consists of a loop over regions. For each region, a loop over the neighboring (external) cells identifies those cells whose removal from their current region would not break that region’s contiguity. For each such cell, the change in the combined spatial and population objective function from reassignment is evaluated, as described in Section 2.1. If, among all possible moves for a region, the best one is an improvement, that cell is removed from the source and added to the destination.

C.4. Metaheuristics. To escape local minima, I have implemented several of the touchstone metaheuristic strategies – greedy, GRASP (Feo and Resende, 1989), and tabu lists (Glover, 1989) – but have found in practice that naïve greedy works. However, I have also implemented a ‘de-stranding’ procedure and a high-level ‘restart’ procedure, that do prove important.

C.4.1. De-stranding. “Strands” or “tentacles” occur somewhat frequently with the inscribed and circumscribing circle methods, where the objective function does not prioritize closer over further cells unless they affect the reference circle. Other algorithms occasionally also fall into a Catch-22 where the end of a long strand is not an “optimal candidate” for a trade, but no cell but the last one can be moved without violating contiguity. Since the last cell is never removed, the strand cannot be removed. I target this behavior directly, by including a configurable “de-strand-ing” procedure that trims away subgraphs connected to the main region at a single node (cell). This method is admittedly ad hoc, but as long strands and singly-connected subgraphs are not consistent with any of the methods near global optima, I believe its sparse use is justifiable.
C.4.2. **Split-restart.** When an optimization does not improve for a configurable number of cycles, it terminates. If activated, the split-restart procedure then (a) splits the district with the worst score on the current metric, and then (b) merges a (random) district with one of its neighbors. This usually results in one district with twice the target population and two with half the target. The interest of this is to give the system a very large shock that nevertheless leaves it in a fairly good starting solution to generate a new partitioning. The shocks are large enough to yield distinct maps in each cycle, but leave the better (more-compact) districts intact. The starting solutions thus typically improve with each cycle. The method is philosophically appealing since it is entirely based on the reference measure; unlike “de-stranding,” it does not import outside prejudices about “good” behavior.

C.5. **Approximations.** Despite significant care to implement the objective functions efficiently, a number of approximations are required to achieve adequate performance. Foremost among these is the use of cell centroids in calculating the convex hull and circumscribing circles, and the various distances (to the perimeter or center). For the circumscribing circles, an approximation of the perimeter is made from the topology, tracing the trijunctions of Census tracts along the border. This method was first suggested by Schwartzberg (1966) for a variant of the IPQ measure. For the fraction of shortest internal paths that are themselves in the district, I constrain the path to run between centroids of adjacent cells, in order to simplify the network problem.

**Appendix D. Spatial, Demographic, and Electoral Data**

This Section briefly reviews the required data. Geographic and demographic data are drawn from various Censuses and the 2015 American Community Survey. The electoral data is substantially less standardized; it leans on both past efforts to assemble precinct-level returns, as well as some data directly from the states.

D.1. **Spatial and Demographic Data.** The fundamental cells for map generation are 2015 Census Tracts, from shapefiles from the Census and tract populations from the 2015 American Community Survey. I have generated topologies from each state geometry using PostGIS 2.1. To reduce perimeter measures’ sensitivity to highly-indented (fractal) perimeter as along waterways, I have simplified the edges of the topology. The simplification is nominally to the 10km level; however if this would result in a new intersection (node/face/edge), the simplification threshold is successively halved until no intersection would result. For optimization and analysis, I project each state into its local EPSG coordinate reference system – usually a Lambert conformal conic or Albers equal area projection on the NAD83(HARN) datum, and always in meters. In states with multiple local projections, I select the center-most one. The list is derived from spatialreference.org, and can be found here.

For historical Congressional Districts, I have used the 107th, 111th, and 114th Congresses, which were drawn after the last three Censuses. Voter ethnicity is similarly drawn from the 1990, 2000, and 2010 Censuses.


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8The exact solution, of crossing a visibility graph is not feasible since the graph must be reconstructed for every proposed move, and the shortest paths reevaluated.
For Pennsylvania in 2012, the precinct names were slightly inconsistent; and a number of manual corrections and fuzzy matches were required. I supplement the Pennsylvania and Texas returns with data directly from the states for Louisiana (2012, 2016), Illinois (2016), Maryland (2016), Minnesota (2008-2016), North Carolina (2012, 2016), Tennessee (2016), Virginia (2016), and Wisconsin (2004-2016). For Maryland in 2016, the polling places and not precincts were available; I therefore use the former. The Virginia shapefiles are courtesy of the Virginia Public Access Project (2017), with some corrections (duplicated layers, and an update for Roanoke City). The Illinois precincts have changed significantly since the 2010 Census release, and I have updated Cook, Dupage, and Lake Counties. Together, these cover most of the changes and more than half of the state’s population. The rest of the state is matched by precinct and county name. In cases like North Carolina where early, absentee, and provisional voting are recorded at the county level, I divide these votes among precincts in proportions equal to the polling-place share of the county vote for each party.

APPENDIX E. A LEGAL HISTORY OF COMPACTNESS

Much of the recent academic, legal, and public discussion of gerrymandering has focussed on remedies through the Supreme Court. I suggest that this focus is misdirected: for much of American history, debates over apportionment and districting centered instead on Congress. It is useful to recall the arc of the historical debates to situate the possible legislative and judicial uses of compactness.

E.1. Congress. The Constitution gives Congress tremendous latitude over the composition of the House and the election of its members. It stipulates that Congress reapportion the House of Representatives after each decennial Census according to the relative populations of the states, but leaves the details of that procedure open. Although authority over the “Times, Places and Manner of holding Elections” is tentatively delegated to state legislatures, Congress retains supervisory power and “may at any time by Law make or alter such Regulations.” Congress is the Judge of its own elections. (U.S. Const., art. 1 § 2, 4, 5)

In the state ratifying conventions, some delegates feared that Congress’s supervisory power over elections might be used to circumvent terms of office. The Federalists defended it on two main grounds. The first was practical and broadly conceded: the Federal government needed direct access to its electors, so as not to be beholden to the state legislatures. In Federalist 59, Hamilton wrote that “every government ought to contain in itself the means of its own preservation.” This was no idle worry: Rhode Island had recently boycotted the Constitutional Convention and would not ratify for another two years. The second justification reflected the same concern as the Guarantee Clause (art. 4, §4): that Congress has a responsibility to ensure just elections and equal representation to its citizens. At the Virginia convention, Madison argued that the citizens of the several states ought to be treated uniformly in the election of their representatives and remarked that “diversity [of regulations] would be obviously unjust.” Of particular interest to the present discussion, he drew for his example the inchoate gerrymander of South Carolina, where Charleston had seized more than its fair share of members. (Kurland, et al., 2000)

While the Founders clearly anticipated that the members of the House would represent geographically defined single-member districts, they did not require it in the Constitution. The ultimate scale of the House – actively defended in Federalist 9 and 10 – was similarly left to Congress in posterity. In discussing the role of the representatives in the House in Federalist 56 and 57, Madison makes it clear that each member was to represent a place-based district
of “thirty thousand inhabitants” with which he would be well-acquainted. New York and Albany were each entitled to one representative, Philadelphia to two, and some of the other Pennsylvania counties to nearly one. Nevertheless, “general ticket” and multi-member districts were commonplace through the country’s history. In the first House elections, Pennsylvania, New Jersey, and Georgia elected members at large (Cong. Quarterly, 1994), and the practice formally ended only in the 91st Congress. (US Code, Title 2, §2c, Amended 1967)

Congress began to flex its powers of self-definition with the Apportionment Act of 1842. Previous Acts had evaluated the proper scale of the House, but the 1842 debate raised the question of the form of its constituencies. It was broadly agreed that single-member districts were preferred in principle, but a number of smaller states had adopted (statewide) “general ticket” elections as a means of ensuring a “bloc vote” that would enhance their power. (Zagarri, 1987) So along with the regular debate over the scale of the House, congressmen sparred over the stipulation that representatives be elected from single-member districts of contiguous territory. Critics of the proposal argued that Congress was either entirely powerless over the shape of districts or else that the supervisory powers of art. 1, §4 gave Congress the power to district – but not to direct the state legislatures how to district. They forecast doom should Congress overstep this bound: it would unleash war between the Federal and state governments. (Cong. Globe, 1842) It is true after the Act passed with the requirement intact, four multi-member states sent members from at-large districts, who were seated. (Cong. Quarterly, 1994) But the Union held, and the states bowed to the new norm within a few years. Though the measure was not reenacted after the 1850 Census, it elicited a tame debate when it was revived and reenacted as a separate (non-apportionment) act in 1862. (Cong. Globe, 1862)

After subsequent Censuses, Congress continued to elaborate its conception of fair representation. In 1872 it added the requirement that districts be equipopulous, which persisted in 1882. (17 Stat. 28, 1872; 22 Stat. 5, 1882) In 1901, an amendment was introduced with the stated goal of avoiding “shoestring” districts, and requiring yet further that the districts be “compact.” (Cong. Rec., 1901) That amendment passed over concerns that the new standard was ill-defined and would prove difficult of application when judging the qualifications of Representatives. The 1911 act reiterated the accumulated requirements: single member districts “composed of a contiguous and compact territory, and containing as nearly as practicable an equal number of inhabitants.” (31 Stat. 733, 1901; 37 Stat. 13, 1911) But following a botched census after the First World War, no act was passed in 1922 and a frantic Congress abandoned the requirements with little debate for the Apportionment Act of 1929. (Cong. Rec., 1929) The Supreme Court ruled in Wood v. Broom (287 U.S. 1, 1932) that Congress’s refusal to reiterate these rules amounted to their annulment.

In the aftermath of the Court’s initial assumption of responsibility for apportionment (below), Congress failed to reassert itself. In 1967 the House and Senate revived the contiguity and compactness debate, and passed a bill requiring them along with equipopulation at the 10% level. The measure failed after its second conference over disagreements on the time allowed to the states for its implementation. In its place, Congress required only that members be elected from single-member districts, which is today the only Federal statute on the form of districts. (Cong. Quarterly, 1968; US Code, Title 2, §2c, Amended 1967)

The interest of these proceedings is that Congress has tremendous powers over the scale of the House and the form of its constituencies – and that it has exercised these powers throughout history. The statutory requirements of compactness, contiguity, and equality of population have been lost, but Congress would be well within the letter and spirit of its constitutional
mandate to revive them. Should it do so, the Court would have very little recourse since under art. 1, §5, “Each House shall be the Judge of the Elections, Returns and Qualifications of its own Members.”

E.2. The Courts. Congress’ retreat was the Court’s ascendancy. This proceeded in two stages: the recognition of the justiciability first of malapportionment and then of political gerrymandering.

E.2.1. The Reapportionment Revolution. Rapid urbanization of the Country through the early 1900s paired with unresponsive state legislators resulted in galling malapportionment of legislative districts by mid-century. After refusing to enter the “political thicket” of apportionment in 1946 with Colegrove v. Green, the Court dramatically revised the division of judicial and legislative authority with Baker v. Carr (369 U.S. 186, 1962). Ruling on a badly-malapportioned Tennessee statehouse, Brennan wrote for the majority that the Court had jurisdiction on equal protection grounds, and further that the question was justiciable and not a “political question” per Colegrove. He classified political questions as those that are inter alia constitutionally committed to another coequal branch of government or characterized by a “lack of judicially discoverable and manageable standards.” Neither condition applied in Baker, and the Court ordered the state to reapportion.

Warren and Brennan had won Stewart’s vote in Baker by convincing him that the decision was narrow, but the “Reapportionment Revolution” was a cataclysm. Justice Whittaker had a nervous breakdown over Baker, recused himself from the case, and retired. Frankfurter suffered a stroke a week after delivering his masterful dissent, and resigned as well. Within two years, the Court formalized its “one person, one vote” standard (Gray v. Sanders, 372 U.S. 368, 1963) and applied it to congressional districts (Wesberry v. Sanders, 376 U.S. 1, 1964) and both houses of state legislatures (Reynolds v. Sims, 377 U.S. 533, 1964). Fighting Reynolds, Senate Minority Leader Everett Dirksen brought the country within one vote of a Constitutional Convention. Reflecting in retirement on a tenure punctuated by Brown and Miranda, Chief Justice Warren called Baker the most consequential decision of his Court. (Smith, 2014)

The Court eliminated malapportionment in just two years, but the success was not unalloyed. With Kirkpatrick v. Preisler (394 U.S. 526, 1969), the Court clarified that the “as nearly as practicable” population equality of Gray and Wesberry was less than 10%. In the majority opinion for Karcher v. Daggett (462 U.S. 725, 1983), Brennan rejected even sub-percent levels of inequality and indeed any de minimis threshold. This had two ill effects. The ‘supremacy’ of the equipopulation requirement was absolute. In his dissent to Baker, Frankfurter argued that the balance of equipopulation with other principles was the very core of the political

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9Frankfurter argued that the definition of the level of equipopulation was a fundamentally political question. Drawing examples from England and the early American states, he argued moreover that it had admitted political solutions. (Baker v. Carr, 369 U.S. 186, 1962, p. 301-310) Second he argued with almost uniquely pertinent originalist sources that the 14th Amendment was not intended to grant Congress power to regulate the suffrage within the states. This argument used the fact that the Joint Committee on the Reconstruction that drafted the 14th Amendment was also charged with restoring the southern states to the Union. This resulted in debates that directly touched on the concern. He argued further that the Union consisted of three categories of states: (i) Union states that approved the Amendment, (ii) Southern states readmitted to the Union under the Joint Committee, and (iii) states that have joined since the Amendment’s ratification. The large majority of these states that did not apportion representatives equally on population, established a legal precedent for unequal apportionment congenital with the Amendment itself. Finally, Frankfurter assessed whether the Court had any say over a state matter (capricious state action), and found that it did not.
problem to be avoided by the Court. The Court ruled at one extreme. Other countries require
looser, 10-40% level of equality, which they balance with other representational principles: mi-
nority representation, preservation of existing subdivisions, etc. (Stephanopoulos, 2013) But
one need not appeal to foreign examples: when Congress debated reasserting its authority
over districts in 1967, the proposed threshold on deviations was similarly just 10%. (Cong.
Quarterly, 1968) Further, the Court’s absolute adherence to equipopulation required fine ma-
nipulation of district lines. It instigated a skyrocketing level of split communities (Altman,
1998), and “normalized” the fingery divisions now familiar in district maps. The second prob-
lem was that after the Court signalled its assumption of responsibility for districting, it has
struggled to deliver beyond apportionment, and Congress has failed to contribute.

E.2.2. Political Gerrymandering. Political gerrymandering, the manipulation of district lines
for partisan advantage, has of late received significant national attention. (Burns and Martin,
2017; McIntee, 2016; Oliver, 2017) Rodden and Chen have conclusively demonstrated that
demographic effects like Democrats’ tendency to cluster in cities currently favors Republic-
ans. Nevertheless, they and others have also shown that a number of states’ maps show persistent,
significant, and growing seat advantages for partisan mapmakers. I reproduce and elaborate
both of these effects in Section 3. In Pennsylvania, Maryland, and North Carolina, the party
controlling the legislature enacted maps for the 114th Congress that yield seats shares for that
party significantly larger than those seen in the compactness-based simulations. Beyond seat
shares, political gerrymandering has been widely demonstrated to reduce trust in government
and voter turn-out. (Stephanopoulos, 2013)

The harms perceived in political gerrymandering are thus real, and the Court since Davis
v. Bandemer (478 U.S. 109, 1986) has held them to be justiciable. However, it has failed to
enunciate a standard for adjudicating it, and therefore declined to provide relief. This is a
curious position, since Brennan’s criterion for justiciability in Baker was the very existence of a
“clear and discoverable standard” for the question at hand. The Court in Davis set a high bar
for such a standard: plaintiffs must show intentional discrimination and actual discriminatory
effect that persistently frustrates a group’s efforts to influence the political process as a whole.

In Davis and each subsequent case before the Court, a slender majority of justices has
tendered proposals or held out hope, and in each case a slender majority has rejected ev-
every standard or justiciability itself. In his dissent (in judgment) to Davis, Powell wrote that
standards from Reynolds ought to be adopted to political gerrymandering. “The most impor-
tant of these factors are the shapes of voting districts and adherence to established political
subdivision boundaries.”

With Vieth v. Jubelirer in 2004 (541 U.S. 267, 2004), four justices argued in favor of relief
according to three standards. Stevens wrote that the “totality of the circumstances” ap-
proach used in racial gerrymandering cases had proven manageable and could be applied to
political gerrymandering. The argument is that racial cases provide an “alternate universe”
where gerrymandering is regulated. Racial gerrymandering is unconstitutional under the 15th
Amendment and regulated under the 1965 Voting Rights Act (as amended in 1975, 1986, and
2006). (US Code §52, 1965) Confronted with this clear charge, the Court has enunciated con-
sistent “traditional districting principles,” of compactness, contiguity, and respect for existing
political subdivisions. The avowed stumbling block in political gerrymandering cases had not
been the harm or the jurisdiction but the standard. The racial gerrymandering cases were
proof positive of a standard. Stevens pointedly remarked, “the plurality does not argue that
the judicially manageable standards that have been used to adjudicate racial gerrymandering claims would not be equally manageable in political gerrymandering cases.” (324)

Souter proposed that a plaintiff who was a member of an identifiable group could make a successful complaint by demonstrating that the group had suffered intentional harm from a plan enacted in disregard of traditional principles. As usual, the principles itemized were “contiguity, compactness, respect for political subdivisions, and conformity with geographic features.” Plaintiffs would have to show that plans that better-respected those principles were possible. (347-351)

For his part, Breyer penned a meditation beginning with “We the People” and seeking “workable form of government that is . . . basically democratic.” He ended by declaiming “unjustified entrenchment . . . purely the result of partisan manipulation and not other factors.” (356, 360, italics original)

Kennedy rejected each of these standards, and went out of his way to reject compactness: “even those criteria that might seem promising at the outset (e.g., contiguity and compactness) are not altogether sound as independent judicial standards for measuring a burden on representational rights. They [...] would unavoidably have significant political effect, whether intended or not.” He nevertheless declined to foreclose hope that a standard could eventually emerge.

Two other themes emerge in the Court’s deliberations in Vieth. The first is that a comfortable majority (the plurality, with Kennedy and Breyer) expressly endorsed politics as a “generally permissible” component of districting. (307) The second theme is the majority’s insistence on a pithy standard. In the plurality opinion, Scalia criticised Powell’s “flabby” approach which Kennedy echoed, calling for a “limited and precise rationale.”

Two years after Vieth, the Court in League of United Latin American Citizens v. Perry (548 U.S. 399, 2006) again failed to identify a standard, but Stevens (joined by Breyer) and Souter (joined by Ginsburg) expressed some interest in developing a standard on partisan symmetry.

The Court’s entry into congressional districting in the Reapportionment Revolution had enormous, immediate, and overwhelmingly positive impact on fair representation in America. Yet its legacy in political gerrymandering has ultimately mirrored Congress’ – a broad recognition of the harm, paired with a toothless consensus over the use of compactness as a component of fair districting laws. Powell’s prophecy for Davis has been fulfilled: it has signalled a “constitutional green light’ to would-be gerrymanderers” while simultaneously “inviting further litigation.”

What is still required – for Congress or the Court – is a “limited and precise” constraint on Congressional districting. Can compactness fill that need?

Appendix F. Compactness as a Legal Standard

The legal history reviewed in Appendix E shows that both Congress and the Court have the potential to eliminate political gerrymandering. Both have recognized compactness as a part of the solution. Section 3 argued that various notions of compactness line up with each other, and that implementing them results in consistent practical effects. Put boldly, there is only one compactness; it is a meaningful standard. Yet the House districts in Figure 6 suggest even without the math, that current House districts are not uniformly compact. Why has compactness failed to gain traction?

Although American history suggests that Congress is at least as well suited to districting reform as the courts, the modern debate has centered on the judiciary. Three broad arguments stand out from that debate, as defenses of the status quo or critiques of compactness:
(1) districting is a legitimate and even quintessentially political activity,
(2) equations are an inadequate palette for expressing political communities, and
(3) the selection of one definition of compactness over another amounts to a political choice.

The first argument – that districting is a political activity – is broadly agreed by the Court. In her concurrence (in judgment) to *Davis*, O’Connor rejected the Court’s grounds and wrote that districting is a “political question in the truest sense.” (*Davis v. Bandemer*, 478 U.S. 109, 1986, p. 145) As remarked above, a majority affirmed this position in *Vieth*. (p. 307) The initial weakness of this argument is that the same defense was marshalled in support of malapportioned districts. The Reapportionment Revolution also recalibrated urban and rural power, and it is the nature of democracy to limit manipulation of the ballot box.

Moreover, inverting the question – asking whether politicians in practice use their authority to express an idealized ‘social value’ – makes a defense of the status quo less credible. The national parties’ activities and avowed motives are partisan, not political, and do not create or elevate cognizable political communities. Unlike town or school district borders, the shifting lines of congressional (let alone state legislator) districts are hardly a standard ‘source of identity.’ Only around half of Americans know the political party of their representatives in Congress (Pew Research Center, 2014), let alone his or her name or the district lines. In a defense of incumbent protections in districting, Nathaniel Persily presents some anecdotes of earnest legislators. He writes that through the districting process, “they create service relationships between representatives and constituents that fit into larger public policy program.” (*Persily*, 2002, p. 32) But his argument is stuck in reverse: in a Democracy this is precisely what is supposed to be what is decided in the ballot box – not the other way around.

The Republican State Leadership Committee’s REDistricting MAjority Project (REDMAP) forwards explicitly partisan goals, which it has executed successfully. Democrats have signalled less-than-beneficent intent to compete on the same terms. (*Burns and Martin*, 2017) In *Coooper v. Harris*, North Carolina Republicans even claimed partisan intent as a defense. And even less-noble justifications like “winner’s bonuses” as for patronage fall flat in the context of elections: entrenchment between elections is anathema to Democracy. The practice is not in short, so high-minded as the principle that the justices appear to defend.

Several authors criticize automation as inadequately nuanced and human. Since it’s impossible to subsume competing conceptions of public interests, communities, and identities in a single pithy metric, any effort is misplaced. Altman writes, “automated redistricting is significantly more complicated than gerrymandering” because the latter only requires “the maximization of one simple goal. Optimal redistricting may involve many simultaneous, complicated, and conflicting goals.” Capturing the “social value” of “subtle patterns of community” in an equation or automated procedure is chasing leprechauns. (*Altman*, 1997, pp. 82, 112) The geometric approach that I have presented is guilty as charged: it does not seek to encode every human interaction and adjacency. The fault of Altman’s argument is that gerrymandering is presently real and pernicious. Progress is possible without perfection; we ought not wallow in a swamp for the impossibility of reaching the end of the rainbow.

The final common critique of gerrymandering is that naming an algorithm is naming a winner. In past papers, the meat of this argument has been that choosing metric or algorithm was simply a re-labelled political choice. Altman wrote, “the idea that automated redistricting is not inherently objective seems both correct and unavoidable.” As cited above, Kennedy wrote in his equivocation for *Vieth* that contiguity and compactness “would unavoidably have significant political effect, whether intended or not.” Since I have shown that a broad array of metrics yields consistent seat shares, the political choice is more constrained than one might
have naïvely believed.\textsuperscript{10} So it is true that the choice of an algorithm will affect electoral outcomes, but the fact that rules result in winners and losers hardly precludes their fairness or neutrality. The question then reduces to compactness’s suitability and fairness. While a criterion of partisan bias might generate different baseline seat shares, the historical and constitutional justification is weak. Compactness on the other hand, has remained central to the debate and woven through the law for over a century. It is a recognized, non-partisan criterion for the principle of geographic representation.

\textbf{Appendix G. The Consistency of Compactness Measures on Historical Maps}

In addition to understanding the political consistency of maps optimized for various measures it is interesting to know whether the various compactness scores or ranks agree on enacted maps. Asking whether a district is compact or not is an important practical question, quite distinct from the possibility of generating compact districts. It could well be that away from the “extrema” chosen by the automated procedures, the compactness measures would rank districts differently. This Section evaluates the consistency of compactness scores and ranks on congressional districts and finds that the spatial “measurements” agree well in practice.

To do this, I turn to the population of maps enacted in the last three districting cycles, in states with more than one representative. I evaluate all of the compactness measures aside path fraction, which has specific technical issues. The power diagrams, split-line procedure, and areal and population distance algorithms (Section 1.2) do not yield scores, and are not included in this analysis. However, I do replace power diagrams with the average interpersonal distance, with the normalization from Section 1.1.9. This gives me 14 measures of compactness (see Figure 3), and 1284 enacted districts.

I perform a principal component analysis (PCA) of this population using Scikit-learn.\textsuperscript{11} (Pedregosa et al., 2011) The explained variance of the first component is 69\% and that of the second component is 14\%. Excluding axis ratio, which has dubious merit, these numbers change to 73\% and 15\%. Renormalizing the measures to have equal variance before the PCA, the loading on the first component falls to 60\% (10\% on the second).

The interpretation is thus again that despite the different shapes shown in Figure 7, compactness in the context of districting reduces to a single concept: it is one-dimensional. The five districts of the 114th Congress that were least compact according to the first principal component are presented in Figure 6. It is notable that of these five, three have already been overturned. The sixth least-compact district is Hawai’i’s second, which encompasses the entire 1500-mile long island chain, save the urban core of Honolulu. That district shows that non-compact districts are not necessarily gerrymandered, but it is the exception that proves the rule.

Figure 7 presents the score and rank correlations among the fourteen measures and the PCA factors. Darker shading (closer to 1) represents higher correlation. As expected, all correlations are positive except between the two components of the PCA in the Spearman’s rank analysis. Axis ratio is notably less-correlated with the other measures. The remaining

\textsuperscript{10}The fact that they give consistent results in automation does not mean that they are equally effective as constraints on gerrymandering. That is, when manipulating lines with partisan intent, it may be easy to keep some compactness scores high and harder for others. This point underscores the strength of a process-based, instead of results-based solution.

\textsuperscript{11}A PCA reduces the dimensionality of a collection of measurements; within a population, it collapses as much of the variation as possible on to a single axis, or component. Of the remaining variation, it collapses as much as possible onto a second axis, and so forth. Each successive component is orthogonal to earlier ones.
Figure 6. Presented are the five districts of the 114th Congress scored least compact by the first component of the compactness principal component analysis (see Section 3). The North Carolina, Florida, and Texas districts have since been struck down.

Figure 7. Correlations among measures of compactness, and with the first two principal components. The correlations are all positive except between the first two primary components in the Spearman’s rank correlation, and so the absolute value is taken.

Thirteen have fairly high score and rank correlations; a poor compactness score or rank by one definition generally predicts a poor score or rank on another definition. By construction, the first principal component is highly correlated with the other measures. It is a good proxy measure of compactness. The correlation with interpersonal distance is particularly high ($\rho = 0.95$); this means that power diagrams will result in good scores across measures. Although the PCA suggests that a large part of compactness is explained by a single axis, I have block-diagonalized the matrices to highlight the various “notions” of compactness.

This analysis suggests that the question of whether or not a real district is compact can meaningfully be answered: it is not just a question of the choice of ruler. From a practical perspective, the first principal component from the analysis represents a robust – though opaque – combined measure of compactness. Optimizing plans for interpersonal distance (or generating power diagrams) will produce districts that score well by all measures.
References for Supporting Materials

An Act for the Apportionment of Representatives to Congress among the several States according to the ninth Census, 42d Congress, 2d session; 17 Stat. 28 (1872).

An act making an apportionment of Representatives in Congress among the several States under the tenth Census, 47th Congress, 1st session; 22 Stat. 5 (1882).

An Act Making an apportionment of Representatives in Congress among the several States under the Twelfth Census, 56th Congress, 2d session; 31 Stat. 733 (1901).

An Act For the apportionment of Representatives in Congress among the several States under the Thirteenth Census, 62d Congress, 1st session; 37 Stat. 13 (1911).


Chou, Christine, Kimbrough, Steven, Sullivan-Fedock, John, Woodard, C. Jason, and Murphy, Frederic H., “Using Interactive Evolutionary Computation (IEC) with Validated Surrogate Fitness Functions for Redistricting,” at 14th Annual Conference


US Code Title 2, §2c (Amended 1967). See the statute amending this section (81 Stat. 581), and Cong. Rec. (1929) for the history of the passage of the 1929 apportionment. The Apportionment Act of 1911 (37 Stat. 13) was the last to contain references to contiguity, compactness, and equality of population.